Integral Homology under Singer–Frobenius Actions

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Abstract

Let X_q be the point–line incidence graph of $\mathrm{PG}(2,q)$ and let $\Gamma_q = C_{q^2+q+1} \rtimes C_3$ be its Singer–Frobenius group. We compute the integral coinvariants $H_1(X_q;\mathbb{Z})_{\Gamma_q}$ and compare them with the ordinary homology of the quotient graph. For $q = 2$ the coinvariants are $\mathbb{Z}/3$, although the quotient graph is a tree. For general q, an integral cycle basis reduces the calculation to a finite-index inclusion of lattices. We then compute the map from invariant cycles to the torsion-free part of the coinvariants. Its Smith factors show that the homology lattice has no Γ_q-stable direct summand isomorphic to the trivial lattice $\mathbb{Z}$. The first case with nonzero invariant cycles, $q = 4$, already displays the integral defect: the comparison map has Smith factors 7 and 63, while the associated norm has a 9-factor. The final section gives a composite-rank obstruction to a prime-to-rank vanishing statement.

1 Introduction

Let $V = \mathbb{F}_q^3$, where $\mathbb{F}_q$ is the field with q elements and q is a prime power. The building considered here is the graph X_q whose vertices are the one- and two-dimensional linear subspaces of V. An edge joins a one-dimensional subspace to a two-dimensional subspace when the former is contained in the latter. Thus X_q is the incidence graph of the projective plane $\mathrm{PG}(2,q)$; it is also the order complex of the nonzero proper subspaces of V. Its top homology provides the integral lattice

$$M_q = \widetilde{H}_1(X_q;\mathbb{Z}) = H_1(X_q;\mathbb{Z}).$$

Here H_1 denotes first homology, and the tilde denotes reduced homology, which agrees with ordinary homology in positive degrees. We shall work directly with cellular chains, as in [3, Section 2.2]. For the classical homological construction of the Steinberg representation, see [9, Section 1.5]. Over a field, the identification with top homology is characteristic-free [10, 11]; Tinberg studies the corresponding Steinberg component for split BN-pairs [12].

Building homology is one source of modules in the study of finite subgroup complexes; see [1, Sections 4–5]. Group actions on these complexes also enter the study of endotrivial modules [2, Section 1]. Here we consider a particular integral lattice and ask how its cyclic splitting changes when Frobenius is included in the acting group. The integral realization matters: a rational Steinberg character can be afforded by different lattices [8, 7, Section 1]. Throughout, M_q means the homology lattice defined above.

For a group H acting on an abelian group A, write

$$A^H = \{a \in A : ha = a \text{ for every } h \in H\}, \qquad A_H = A/\langle ha - a : h \in H,\ a \in A\rangle$$

for the invariants and coinvariants, respectively. The brackets in the second formula mean the additive subgroup generated by the displayed differences. Thus coinvariants identify elements in the same orbit, but they need not preserve kernels of boundary maps. This is the source of the integral effect studied below: the graph quotient has ordinary homology, whereas the coinvariants of the homology lattice may retain torsion.

There are two stages in the calculation. The Singer subgroup K acts freely on the graph, so the quotient $Y = X_q/K$ is a graph with two vertices and $q+1$ edges. We compare the lattice $(M_q)_K$ with $H_1(Y;\mathbb{Z})$ and identify the finite quotient between them. Frobenius then acts on this small quotient graph by a permutation of its edges. The three congruence classes of q arise from these two pieces of integral data: the number of Frobenius-fixed edges and the multiplier q on the cyclic quotient $\mathbb{Z}/m$.

Our main family calculation is the following.

Theorem 1.1. *Let q be a prime power, put $m = q^2+q+1$, and realize V as $\mathbb{F}_{q^3}$ over $\mathbb{F}_q$. Let $K = \mathbb{F}_{q^3}^\times/\mathbb{F}_q^\times$ act by multiplication on projective points and lines. Let σ be induced by $x \mapsto x^q$, and let $\Gamma_q = K \rtimes \langle\sigma\rangle$, where $\langle\sigma\rangle \cong C_3$. Then*

$$(M_q)_{\Gamma_q} \cong \begin{cases} \mathbb{Z}^{q/3}, & q \equiv 0 \pmod 3, \\ \mathbb{Z}^{(q+2)/3}, & q \equiv 1 \pmod 3, \\ \mathbb{Z}^{(q-2)/3} \oplus \mathbb{Z}/3, & q \equiv 2 \pmod 3. \end{cases}$$

Here C_r denotes a cyclic group of order r. The multiplication subgroup $K \cong C_m$ is a Singer subgroup. In addition,

$$M_q\big|_K \cong \mathbb{Z} \oplus \mathbb{Z}[K]^{q-1},$$

where $\mathbb{Z}[K]$ is the integral group ring, the vertical bar denotes restriction of the action to K, and the action on the first summand is trivial.

We start with $q = 2$ to fix the integral conventions. The same chain calculation then works for every prime power. The cyclic projective-plane construction is classical [5]; the calculations below retain the integral lattice maps instead of replacing them by character computations.

2 The Fano graph and a normalizer of order twenty-one

All subscripts in this section lie in $\mathbb{Z}/7$, so that, for example, $P_7 = P_0$. Set $D = \{0,1,3\}$, the first difference set in [5, p. 384]. Define a bipartite graph X with vertices P_i, L_i for $i \in \mathbb{Z}/7$ and oriented edges

$$e_{i,d} : P_i \longrightarrow L_{i+d} \qquad (d \in D).$$

The orientation is a convention for chains; the underlying graph is unoriented. There are fourteen vertices and twenty-one edges. The edges with $d = 0, 1$ already form a connected fourteen-cycle.

The six ordered differences of distinct elements of D are exactly the six nonzero residues modulo seven. Consequently two distinct points have exactly one common line. This is the cyclic incidence model of the Fano plane. Identify it with the subspace building as follows.

Let α be the image of u in $\mathbb{F}_8 = \mathbb{F}_2[u]/(u^3+u^2+1)$. The polynomial has no root in $\mathbb{F}_2$ and is therefore irreducible. The element α generates $\mathbb{F}_8^\times$, which has order seven. Define

$$\mathrm{Tr}(x) = x + x^2 + x^4, \qquad P_i = \mathbb{F}_2\alpha^i, \qquad L_j = \ker\big(x \mapsto \mathrm{Tr}(\alpha^{6-j}x)\big).$$

The trace takes values in $\mathbb{F}_2$. The seven displayed kernels are the seven distinct two-dimensional subspaces: each functional is nonzero, and two of them cannot agree, since for $a \neq b$ the functional $x \mapsto \mathrm{Tr}((a-b)x)$ takes the value $\mathrm{Tr}(1) = 1$ at $x = (a-b)^{-1}$. Using $\alpha^3 = \alpha^2 + 1$ gives

i	0	1	2	3	4	5	6
$\mathrm{Tr}(\alpha^i)$	1	1	1	0	1	0	0

and hence $P_i \subset L_j$ if and only if $i+6-j \in \{3,5,6\}$, or equivalently $j-i \in D$. This proves $X = X_2$.

Define two automorphisms of the graph by

$$t(P_i) = P_{i+1}, \qquad t(L_i) = L_{i+1}, \tag{2.1}$$
$$\sigma(P_i) = P_{2i}, \qquad \sigma(L_i) = L_{2i+1}. \tag{2.2}$$

Multiplication by α gives the first, and Frobenius $x \mapsto x^2$ gives the second. Incidence is preserved because $2D + 1 = D$, with the permutation $0 \mapsto 1 \mapsto 3 \mapsto 0$. They satisfy

$$t^7 = 1, \qquad \sigma^3 = 1, \qquad \sigma t \sigma^{-1} = t^2.$$

Thus $\Gamma = \langle t, \sigma \rangle \cong C_7 \rtimes C_3$ has order twenty-one. The cyclic action is regular, meaning free and transitive, on both types of vertex.

This Γ is the full normalizer of $\langle t \rangle$ in $\mathrm{GL}_3(\mathbb{F}_2)$. Indeed a linear map commuting with multiplication by α commutes with every element of $\mathbb{F}_2[\alpha] = \mathbb{F}_8$, so its invertible centralizer consists of the seven nonzero multiplication maps. If a linear map conjugates t to t^k, then α^k must have the same minimal polynomial as α. Its roots are $\alpha, \alpha^2, \alpha^4$, so $k \in \{1, 2, 4\}$. Frobenius realizes these three possibilities.

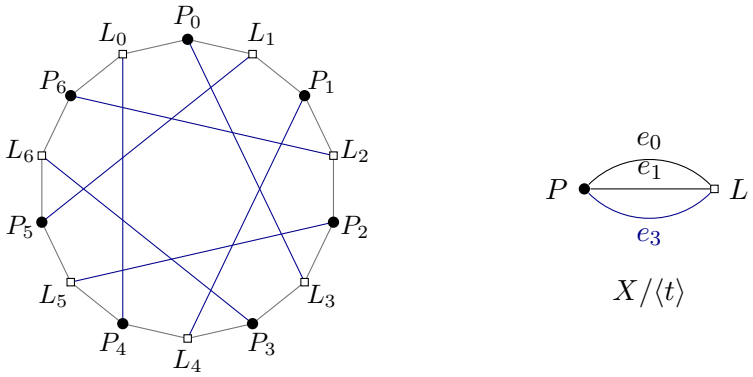


Figure 1: The Fano incidence graph and its cyclic quotient. Dots are points and squares are lines. The outer fourteen-cycle represents w; the blue chords have $d = 3$. The quotient has two vertices and three edges. Edge orientations are fixed by the formulas in the text.

3 An explicit integral cycle basis

Put

$$R = \mathbb{Z}[t]/(t^7 - 1) = \mathbb{Z}[C_7], \qquad \nu = 1 + t + \cdots + t^6, \qquad S = 1 + t + t^2.$$

The variable t acts by the graph automorphism (2.1); R is a ring of finite integral linear combinations of its powers. Write $e_d = e_{0,d}$, $a = P_0$, and $b = L_0$ for the chain generators. Since the cyclic action is free on vertices and edges, the cellular groups and boundary are

$$C_1(X;\mathbb{Z}) = Re_0 \oplus Re_1 \oplus Re_3, \quad C_0(X;\mathbb{Z}) = Ra \oplus Rb, \quad \partial e_d = t^d b - a.$$

There are no two-dimensional cells, so $M = H_1(X;\mathbb{Z}) = \ker \partial$.

Lemma 3.1. *The annihilator* $\{r \in R : (t-1)r = 0\}$ *of* $t - 1$ *is* $\mathbb{Z}\nu$.

Proof. Write $r = \sum_{i=0}^{6} r_i t^i$. The coefficient of t^i in $(t-1)r$ is $r_{i-1} - r_i$, with subscripts modulo seven. It vanishes for every i precisely when all seven coefficients are equal. □

Proposition 3.2. *The cycles*

$$w = \nu(e_1 - e_0), \qquad z = e_3 - Se_1 + (S-1)e_0$$

give an isomorphism of R-modules

$$\mathbb{Z} \oplus R \longrightarrow M, \qquad (c, r) \longmapsto cw + rz,$$

where t acts trivially on $\mathbb{Z}$. In particular $w, z, tz, \dots, t^6 z$ form an integral basis of M.

Proof. A chain $x_0e_0 + x_1e_1 + x_3e_3$ is a cycle precisely when

$$x_0 + x_1 + x_3 = 0, \qquad x_0 + tx_1 + t^3x_3 = 0.$$

Eliminating x_0 and using $t^3 - 1 = (t-1)S$ gives $(t-1)(x_1 + Sx_3) = 0$. Lemma 3.1 gives a unique integer c such that $x_1 = c\nu - Sx_3$, and then $x_0 = -c\nu + (S-1)x_3$. This is exactly $cw + x_3z$. Uniqueness also follows directly from the coefficient of e_3 and the fact that ν is a nonzero element of the free abelian group R. Finally $t\nu = \nu$, so $tw = w$. □

The chain z is the oriented hexagon

$$P_0 \longrightarrow L_3 \longrightarrow P_2 \longrightarrow L_2 \longrightarrow P_1 \longrightarrow L_1 \longrightarrow P_0.$$

The signs in its formula record the orientation of each traversal. The chain w travels around the fourteen-cycle using an edge of type 1 followed by the reverse of an edge of type 0. Thus the basis has a geometric interpretation: one long cycle and seven translates of one hexagon. Its rank is eight, agreeing with $21 - 14 + 1$, the first Betti number of a connected graph.

4 Frobenius mixes the two cyclic summands

The Frobenius action is semilinear with respect to the automorphism $t \mapsto t^2$ of R:

$$\sigma(rv) = r(t^2)\sigma(v).$$

On the edge generators it gives $\sigma(e_0) = e_1$, $\sigma(e_1) = e_3$, $\sigma(e_3) = e_0$. Put $A = 1 + t^2 + t^4$. Since $\sigma(\nu) = \nu$,

$$\sigma(w) = 2w + \nu z, \tag{4.1}$$

$$\sigma(z) = -w - Az. \tag{4.2}$$

To verify the first formula, note that $\nu z = \nu e_3 - 3\nu e_1 + 2\nu e_0$. For the second, substitute $e_3 = z + Se_1 - (S-1)e_0$ in $\sigma(z) = e_0 - Ae_3 + (t^2 + t^4)e_1$ and use the polynomial identities

$$AS - (t^2 + t^4) = \nu, \qquad A(S-1) + 1 = \nu.$$

These computations take place over the integers; no averaging or division by seven is used.

Proposition 4.1. *The lattice M has no nonzero Γ-fixed element and no Γ-stable direct summand isomorphic to the trivial lattice $\mathbb{Z}$.*

Proof. By Proposition 3.2 and Lemma 3.1, $M^{C_7} = \mathbb{Z}w \oplus \mathbb{Z}\nu z$. On this basis σ has matrix

$$\begin{pmatrix} 2 & -7 \\ 1 & -3 \end{pmatrix}.$$

An invariant vector $aw + b\nu z$ must satisfy $a = 7b$ and $a = 4b$, hence $a = b = 0$. An action on a free rank-one abelian group takes values in $\{1, -1\}$. Since $|\Gamma| = 21$ is odd, such an action must be trivial. A stable summand isomorphic to the trivial lattice $\mathbb{Z}$ would therefore provide a nonzero fixed element, which has just been excluded. □

Thus the C_7-stable summand $\mathbb{Z}w$ cannot be replaced by a Γ-stable trivial summand. Section 7 proves the same non-splitting assertion for every q, including cases in which there are nonzero Γ_q-invariant cycles.

5 Coinvariants, torsion, and the ordinary quotient

Proposition 5.1. $M_\Gamma \cong \mathbb{Z}/3$. *However,* $H_1(X/\Gamma;\mathbb{Z}) = 0$.

Proof. Take C_7-coinvariants first. By Proposition 3.2, M_{C_7} is free on the images $\bar{w}$, $\bar{z}$, and νz becomes $7\bar{z}$. The bar denotes the class in this quotient. Since C_7 is normal, σ acts on it, with matrix

$$B = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}.$$

Writing I_2 for the two-by-two identity matrix, taking the remaining coinvariants means quotienting by the image of $B - I_2$. This imposes the two relations

$$\bar{w} + 7\bar{z} = 0, \qquad -\bar{w} - 4\bar{z} = 0.$$

Eliminating $\bar{w}$ leaves exactly $3\bar{z} = 0$. Equivalently $B - I_2$ has determinant 3 and entries of greatest common divisor 1, so its Smith normal form has diagonal $1, 3$.

The action on X has one orbit of points, one orbit of lines, and one orbit of edges: t moves the first edge subscript and σ cycles the three values of d. It never exchanges vertex types. The ordinary quotient is therefore one edge joining two vertices, so it has zero first homology. □

The homology group M itself is torsion-free. The three-torsion appears only after imposing the group-action relations. More concretely,

$$H_1\big(C_*(X;\mathbb{Z})_\Gamma\big) = 0, \qquad H_1\big(C_*(X;\mathbb{Z})\big)_\Gamma = \mathbb{Z}/3.$$

Here C_* denotes the cellular chain complex and its subscript denotes degreewise coinvariants. The first complex has boundary $\mathbb{Z} \to \mathbb{Z}^2$, $1 \mapsto (-1, 1)$, which is injective. The second construction takes the kernel before taking coinvariants.

This gives a concrete test for the loss of information. For a field k, homology of a graph commutes with extension of coefficients, and coinvariants commute with tensor products. Hence

$$H_1(X;k)_\Gamma \cong (\mathbb{Z}/3) \otimes_{\mathbb{Z}} k = \begin{cases} k, & \operatorname{char} k = 3, \\ 0, & \operatorname{char} k \neq 3. \end{cases}$$

In particular the rational calculation misses the torsion entirely. Likewise it vanishes after localization at the defining prime 2; the integral phenomenon here is consistent with the localized coefficient conventions in [1, Remark 2.4].

The rational character gives an independent check:

element type	1	t^i $(i \neq 0)$	$t^i\sigma^{\pm 1}$
number of elements	1	6	14
trace on $M \otimes \mathbb{Q}$	8	1	-1

The cyclic splitting gives the middle column. An element in either Frobenius coset fixes one point and one line but no edge, giving the last column from $H_0 - C_0 + C_1$. Averaging the traces yields zero, agreeing with Proposition 4.1 over $\mathbb{Q}$.

6 The family calculation

6.1 Cyclic coordinates and an integral lattice inclusion

Set $m = q^2 + q + 1$. A cyclic difference set for the projective plane is a subset $D \subset \mathbb{Z}/m$ of size $q+1$ such that every nonzero residue occurs exactly once as an ordered difference of distinct elements of D. The Singer construction supplies such a set [5]; see also [6, Section 10.2.1] for cyclic coordinates. Equivalently, label the points and lines by the two regular K-orbits and record which lines contain P_0. Uniqueness of the line through two points gives the difference-set condition. Translating D allows us to assume $0, 1 \in D$: choose the ordered pair whose difference is 1. This is Singer's reduced difference-set normalization [5, p. 382].

As before there is an edge $e_{i,d} : P_i \to L_{i+d}$. Put

$$R_m = \mathbb{Z}[t]/(t^m - 1), \qquad \nu_m = \sum_{i=0}^{m-1} t^i, \qquad S_d = \sum_{i=0}^{d-1} t^i \quad (1 \le d < m),$$

choosing the representatives of D in $\{0, \dots, m-1\}$. The boundary is still $\partial e_d = t^d b - a$. The same elimination as in Proposition 3.2 gives

$$M_q = \mathbb{Z}w \oplus \bigoplus_{d \in D \setminus \{0,1\}} R_m z_d, \quad w = \nu_m(e_1 - e_0), \quad z_d = e_d - S_d e_1 + (S_d - 1)e_0. \tag{6.1}$$

Indeed the remaining boundary equation is

$$(t-1)\left(x_1 + \sum_{d \in D \setminus \{0,1\}} S_d x_d\right) = 0,$$

whose parenthesis is a unique integer multiple of ν_m. This proves the cyclic splitting of Theorem 1.1. The rank is $1 + (q-1)m = q^3$; this also follows directly from the numbers of vertices and edges of the incidence graph.

Let $Y = X_q/K$. It has two vertices and $q+1$ edges. Write $L = H_1(Y;\mathbb{Z})$; the letter L now denotes a lattice, rather than a projective line L_i. It has basis $c_d = e_d - e_0$ for $d \in D \setminus \{0\}$. Projection of cycles induces an injective map

$$\begin{aligned} \iota : (M_q)_K &\longrightarrow L, \\ \bar{w} &\longmapsto mc_1, \qquad\qquad \bar{z}_d \longmapsto c_d - dc_1. \end{aligned}$$

The matrix of this map is triangular with determinant m. Its image is precisely the kernel of the surjective map

$$\rho : L \longrightarrow \mathbb{Z}/m, \qquad \rho(c_d) = d \pmod m.$$

This follows by eliminating the coordinates of the c_d with $d \neq 1$; the remaining coefficient of c_1 is divisible by m exactly when ρ vanishes. Thus the cycle calculation gives the exact sequence

$$0 \longrightarrow (M_q)_K \xrightarrow{\iota} L \xrightarrow{\rho} \mathbb{Z}/m \longrightarrow 0. \tag{6.2}$$

The degree-one part of the Cartan–Leray sequence for the free K-action is

$$H_2(K;\mathbb{Z}) \longrightarrow (M_q)_K \longrightarrow H_1(Y;\mathbb{Z}) \longrightarrow H_1(K;\mathbb{Z}) \longrightarrow 0;$$

see [4, Section 5.2, formula (5.5)]. For $K = C_m$, the outer group on the left is zero and $H_1(K;\mathbb{Z}) = \mathbb{Z}/m$ [4, Example 3.2]. The cycle basis above identifies the maps in this standard sequence.

6.2 The Frobenius permutation

In the finite-field model of Theorem 1.1, Frobenius fixes $P_0 = \mathbb{F}_q 1$. Hence its action on the edges of Y is identified with its action on the $q+1$ projective lines through P_0. The number f of fixed lines is

$$f = \begin{cases} 1, & q \equiv 0 \pmod 3, \\ 2, & q \equiv 1 \pmod 3, \\ 0, & q \equiv 2 \pmod 3. \end{cases} \tag{6.3}$$

To prove this, let $F : x \mapsto x^q$ act on the three-dimensional $\mathbb{F}_q$-space $\mathbb{F}_{q^3}$. Its fixed space is exactly $\mathbb{F}_q 1$ and $F^3 = 1$. When $q \equiv 2 \pmod 3$, the characteristic is not 3, so the factors $T-1$ and T^2+T+1 are relatively prime, and the action on $\mathbb{F}_{q^3}/\mathbb{F}_q 1$ has minimal polynomial T^2+T+1. This polynomial is irreducible over $\mathbb{F}_q$, because $\mathbb{F}_q^\times$ has no nontrivial cube root of unity when $q \equiv 2 \pmod 3$. A fixed projective line through P_0 would give an invariant one-dimensional subspace of that quotient, which is impossible.

When $q \equiv 1 \pmod 3$, choose a nontrivial cube root of unity $\omega \in \mathbb{F}_q$. Each of the equations $x^q = \omega x$ and $x^q = \omega^2 x$ has a nonzero solution in $\mathbb{F}_{q^3}$: the subgroup of $(q-1)$st powers in $\mathbb{F}_{q^3}^\times$ has order m, which is divisible by 3. Each solution set is a one-dimensional eigenspace over $\mathbb{F}_q$. Together with $\mathbb{F}_q 1$ these give the three distinct eigenspaces of F. There are exactly two invariant lines in the quotient by $\mathbb{F}_q 1$, hence exactly two fixed projective lines through P_0.

Finally, when the characteristic is 3, $(F-1)^3 = 0$ and $\dim \ker(F-1) = 1$. The Jordan form is therefore one block of size three. The quotient by its fixed line has one block of size two and exactly one invariant one-dimensional subspace. This proves the remaining case of (6.3).

Thus the edges of Y consist of f fixed edges and $r = (q+1-f)/3$ free orbits of $C_3 = \langle \sigma \rangle$. The lattice L is the augmentation kernel of their permutation module:

$$L = \ker\bigl(\text{aug} : \mathbb{Z}^f \oplus \mathbb{Z}[C_3]^r \longrightarrow \mathbb{Z}\bigr). \tag{6.4}$$

The augmentation aug adds all the coefficients; this is exactly the graph-cycle condition that the oriented edge coefficients sum to zero.

Frobenius also acts on the last term of (6.2) by multiplication by q. The underlying multiplier property is classical [5, pp. 382–383]. One can check the induced map in cyclic coordinates: the edge-label permutation has form $\pi(d) = qd + c \pmod m$ for one constant c, and $\sigma(c_d) = c_{\pi(d)} - c_{\pi(0)}$, where $c_0 = 0$. It follows that $\rho(\sigma c_d) = qd = q\rho(c_d)$ modulo m.

Lemma 6.1. *If $q \equiv 0$ or $2 \pmod 3$, the inclusion ι induces an isomorphism $((M_q)_K)_{C_3} \cong L_{C_3}$.*

Proof. We have $\gcd(q-1, m) = \gcd(q-1, 3) = 1$, so $\sigma - 1$ is invertible on the quotient $\mathbb{Z}/m$ in (6.2). For surjectivity on coinvariants, let $y \in L$. Choose $u \in L$ whose image in $\mathbb{Z}/m$ maps to $\rho(y)$ under $\sigma - 1$. Then $y - (\sigma-1)u \in \iota((M_q)_K)$, so its class equals that of y. For injectivity, suppose $x \in \iota((M_q)_K)$ and $x = (\sigma-1)y$ for some $y \in L$. Its image in $\mathbb{Z}/m$ is zero; invertibility forces $\rho(y) = 0$. Hence y already lies in the image of ι, giving the required relation inside that image. □

6.3 The three cases and the projection map

Suppose first $q \equiv 2 \pmod 3$, so $f = 0$ and $r = (q+1)/3$. Put $I = \ker(\text{aug} : \mathbb{Z}[C_3] \to \mathbb{Z})$. An explicit module isomorphism is

$$\mathbb{Z}[C_3]^{r-1} \oplus I \longrightarrow L, \qquad (x_1, \dots, x_{r-1}, y) \longmapsto \left(x_1, \dots, x_{r-1}, y - \sum_{j=1}^{r-1} x_j \right).$$

It is invertible by adding the first $r-1$ coordinates to the last. The two elements $a=\sigma-1$ and $b=\sigma^2-\sigma$ form an integral basis of I. Their action is $\sigma a=b$ and $\sigma b=-a-b$. Taking coinvariants gives $a=b$ and $3a=0$, with no further relation. Therefore

$$L_{C_3}\cong\mathbb{Z}^{r-1}\oplus I_{C_3}\cong\mathbb{Z}^{r-1}\oplus\mathbb{Z}/3.$$

Taking successive coinvariants by a normal subgroup and its quotient is the same as imposing all the group relations at once. Combining this observation with Lemma 6.1 proves the $q\equiv 2 \pmod 3$ case.

When $f\geq 1$, choose one fixed edge as the augmentation anchor. Eliminating its coefficient in (6.4) gives

$$L\cong\mathbb{Z}^{f-1}\oplus\mathbb{Z}[C_3]^r,\qquad L_{C_3}\cong\mathbb{Z}^{f+r-1}.$$

For $q\equiv 0 \pmod 3$ we have $f=1$ and $r=q/3$; Lemma 6.1 gives the first case of the theorem.

It remains to handle $q\equiv 1 \pmod 3$, where $f=2$ and $\gcd(q-1,m)=3$. The fixed subgroup of the quotient $\mathbb{Z}/m$ in (6.2) is cyclic of order three. Let d,e be the two fixed edge labels. The cycle $c_d-c_e\in L$ is fixed by σ; as above $c_0=0$ if one of the labels is zero. Its image under ρ is the nonzero residue $d-e$, which lies in $\ker(q-1)$ because $qd+c=d$ and $qe+c=e$. It therefore generates that subgroup of order three. In particular every invariant element of the finite quotient has an invariant lift to L.

This lifting property implies that $((M_q)_K)_{C_3}\to L_{C_3}$ is injective. Indeed, if $x=(\sigma-1)y$ belongs to $\iota((M_q)_K)$, then $\rho(y)$ is fixed. Choose a fixed $h\in L$ with $\rho(h)=\rho(y)$. Now $y-h$ lies in the image of ι and $x=(\sigma-1)(y-h)$ is already a relation there. The cokernel is $(\mathbb{Z}/m)_{C_3}=\mathbb{Z}/3$, since imposing the action relation on $\mathbb{Z}/m$ is quotienting by multiplication by $q-1$. Thus $((M_q)_K)_{C_3}$ is a subgroup of index three in the free lattice $L_{C_3}\cong\mathbb{Z}^{r+1}$. It is itself free of that rank, namely $r+1=(q+2)/3$. This completes the proof of Theorem 1.1.

Projection induces the natural map

$$\eta_q:(M_q)_{\Gamma_q}\longrightarrow H_1(X_q/\Gamma_q;\mathbb{Z}).$$

The ordinary quotient has two vertices and $f+r$ edges. For $f\geq 1$ the augmentation-anchor description above identifies L_{C_3} with its first homology. For $f=0$, the description $L\cong\mathbb{Z}[C_3]^{r-1}\oplus I$ shows that projection kills the $I_{C_3}=\mathbb{Z}/3$ summand and maps the remaining summands isomorphically onto the quotient's cycle lattice. Consequently

$q \bmod 3$	0	1	2
$\ker\eta_q$	0	0	$\mathbb{Z}/3$
$\operatorname{coker}\eta_q$	0	$\mathbb{Z}/3$	0

The two occurrences of three-torsion have different meanings: for $q\equiv 2$ they are torsion in the source, while for $q\equiv 1$ they measure a finite index between two torsion-free lattices.

To distinguish these effects, define

$$\delta_q=\operatorname{coker}\big(((M_q)_K)_{C_3}\longrightarrow L_{C_3}\big),$$

then the calculation above gives

$$\delta_q\cong\begin{cases}0, & q\equiv 0 \pmod 3,\\ \mathbb{Z}/3, & q\equiv 1 \pmod 3,\\ 0, & q\equiv 2 \pmod 3.\end{cases}\qquad \operatorname{tors}(M_{q\Gamma_q})\cong\begin{cases}0, & q\equiv 0,1 \pmod 3,\\ \mathbb{Z}/3, & q\equiv 2 \pmod 3.\end{cases}$$

The first table records a finite index between free lattices; the second records torsion created by taking coinvariants. The two groups must be kept separate in the Smith-form calculation.

q	m	rank M_q	r	$(M_q)_{\Gamma_q}$
2	7	8	1	$\mathbb{Z}/3$
3	13	27	1	$\mathbb{Z}$
4	21	64	1	$\mathbb{Z}^2$
5	31	125	2	$\mathbb{Z} \oplus \mathbb{Z}/3$
8	73	512	3	$\mathbb{Z}^2 \oplus \mathbb{Z}/3$
11	133	1331	4	$\mathbb{Z}^3 \oplus \mathbb{Z}/3$

These are consequences of the proof, not a pattern inferred from a finite list of computations. The case distinction comes from both the Frobenius fixed lines and the action on the finite quotient in (6.2); keeping only one of these ingredients would miss the index-three phenomenon.

7 Invariant cycles and integral splitting

Nonzero invariant cycles need not give an equivariant trivial direct summand. To decide this, one must also examine invariant integral functionals. Put

$$Q_q = (M_q)_{\Gamma_q}/\operatorname{tors}((M_q)_{\Gamma_q}), \qquad j_q : M_q^{\Gamma_q} \longrightarrow Q_q,$$

where $\operatorname{tors}(A)$ denotes the elements of finite order in an abelian group A, and j_q is inclusion followed by projection. We compute the Smith invariant factors of this map, that is, the diagonal entries in its Smith normal form.

Theorem 7.1. *The map j_q is injective with finite cokernel. With $m = q^2 + q + 1$,*

$$\operatorname{coker} j_q \cong \begin{cases} (\mathbb{Z}/(3m))^{q/3}, & q \equiv 0 \pmod 3, \\ \mathbb{Z}/(m/3) \oplus (\mathbb{Z}/(3m))^{(q-1)/3}, & q \equiv 1 \pmod 3, \\ (\mathbb{Z}/(3m))^{(q-2)/3}, & q \equiv 2 \pmod 3. \end{cases}$$

An exponent zero denotes the zero group. In particular, M_q has no Γ_q-stable direct summand isomorphic to the trivial lattice $\mathbb{Z}$.

Proof. Retain $L = H_1(X_q/K;\mathbb{Z})$ and the numbers f, r from Section 6. Summing all the lifts of each edge defines the transfer

$$T : L \longrightarrow M_q^K, \qquad T(e_d - e_e) = \nu_m(e_d - e_e).$$

Every K-invariant chain has constant coefficients on each edge orbit. Its boundary vanishes exactly when those coefficients sum to zero. Thus T is an isomorphism onto M_q^K. It commutes with Frobenius, since Frobenius permutes the lifts, and hence identifies L^{C_3} with $M_q^{\Gamma_q}$. Let $p : M_q \to L$ be induced by projection back to X_q/K. Then $pT = m\,\mathrm{id}_L$, the usual covering-transfer identity [3, Section 3.G].

For $q \equiv 0, 2 \pmod 3$, Lemma 6.1 identifies the coinvariants of M_q with L_{C_3}. Under this identification, the class of $T(y)$ is $m\bar{y}$. If $q \equiv 0$, the fixed-edge anchor gives $L \cong \mathbb{Z}[C_3]^r$. Each invariant generator is the sum of its three translates, so its image in coinvariants is three times the corresponding generator. The matrix of j_q is $3mI_r$, where I_r denotes the r-by-r identity matrix.

If $q \equiv 2$, use $L \cong \mathbb{Z}[C_3]^{r-1} \oplus I$. The augmentation ideal I has no nonzero invariant element: an invariant in $\mathbb{Z}[C_3]$ has equal coefficients, and its augmentation is three times that coefficient. On the other summands the comparison map is again multiplication by three. After discarding $I_{C_3} = \mathbb{Z}/3$, the matrix of j_q is $3mI_{r-1}$. These matrices prove the first and third cases.

Suppose now $q \equiv 1 \pmod 3$. Choose the fixed edges e_d, e_e, use e_e as anchor, and put $h = e_d - e_e$. Choose one edge from each free orbit, and write $x_a = e_a - e_e$ for its cycle class in L_{C_3}. Then

$$L_{C_3} = \mathbb{Z}h \oplus \bigoplus_a \mathbb{Z}x_a.$$

The image N of $(M_q)_{\Gamma_q}$ is the kernel of

$$\psi : L_{C_3} \longrightarrow \mathbb{Z}/3, \qquad \psi(\bar{y}) = \rho(y) \pmod 3.$$

This follows by taking coinvariants in (6.2). The difference $d - e$ is a nonzero element of order three in $\mathbb{Z}/m$, so it equals $m/3$ or $2m/3$ there. Writing $q = 1 + 3k$ gives $m/3 = 1 + 3k + 3k^2 \equiv 1 \pmod 3$. Consequently $\psi(h)$ is a unit in $\mathbb{Z}/3$.

For each a, choose $b_a \in \{0, 1, 2\}$ with $\psi(x_a) = b_a\psi(h)$. An integral basis of N is

$$h_0 = 3h, \qquad x'_a = x_a - b_a h.$$

An integral basis of L^{C_3} consists of h and the cycles $y_a = e_a + \sigma e_a + \sigma^2 e_a - 3e_e$. Their classes in L_{C_3} are h and $3x_a$, respectively. Since $pT = m$, the matrix of j_q in these bases has first column $(m/3, 0, \dots, 0)^T$ and column a equal to $(b_a m, 0, \dots, 3m, \dots, 0)^T$. Subtracting $3b_a$ times the first column from column a gives

$$\operatorname{diag}(m/3, 3m, \dots, 3m).$$

The first entry divides all the others. This is the Smith normal form and proves the second case, as well as injectivity in every case.

Finally suppose $M_q = \mathbb{Z}v \oplus W$ with both summands Γ_q-stable. The order $3m$ of Γ_q is odd, so its action on $\mathbb{Z}v$ must be trivial. The summand projection gives a Γ_q-invariant homomorphism $\lambda : M_q \to \mathbb{Z}$ with $\lambda(v) = 1$. Such a homomorphism factors through Q_q. But all elements in the image of j_q are divisible in Q_q by $3m$ when $q \equiv 0, 2$, and by $m/3$ when $q \equiv 1$. Each divisor is greater than one; in the second case $q \geq 4$. Thus $\lambda(v) = 1$ is impossible. For $q = 2$ the invariant group is already zero. □

For example, when $q = 4$ the invariant group and Q_q both have rank two, but j_q has diagonal entries $7, 63$. When $q = 5$ they both have rank one and j_q is multiplication by 93. The absence of an equivariant trivial direct summand in these examples follows from the integral index, rather than from a vanishing invariant space. The distinction is essential: a primitive fixed vector can span a direct summand of the underlying abelian group while admitting no stable complement.

7.1 The first nontrivial case: $q = 4$

The case $q = 4$ is the smallest example in which invariant cycles exist and the comparison map is nevertheless non-primitive. Here $m = 21$, and one reduced Singer difference set is

$$D = \{0, 1, 4, 14, 16\} \subset \mathbb{Z}/21.$$

Frobenius acts by multiplication by 4 on the labels. Thus 0 and 14 are fixed, while $1, 4, 16$ form a three-cycle. The quotient lattice $L = H_1(X_4/K; \mathbb{Z})$ therefore has rank four, and

$$L_{C_3} \cong \mathbb{Z}^2, \qquad (M_4)_{\Gamma_4} \cong \mathbb{Z}^2.$$

The projection from coinvariants to the homology of the full quotient has cokernel $\mathbb{Z}/3$: the two lattices have the same rank, but the image of the former has index three in the latter. The invariant-cycle comparison map has Smith form

$$j_4 \sim \begin{pmatrix} 7 & 0 \\ 0 & 63 \end{pmatrix}.$$

Since $|\Gamma_4| = 63$, the corresponding norm has Smith form

$$N \sim \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives $\widehat{H}^0(\Gamma_4, M_4) \cong \mathbb{Z}/9$. The factor 9 is the quotient $|\Gamma_4|/(m/3) = 63/7$ forced by the exceptional Smith factor in j_4 and the identity $Nj_4 = |\Gamma_4|\,\mathrm{id}$.

Over $\mathbb{Q}$, the map $j_q \otimes \mathbb{Q}$ is an isomorphism. Thus the rational Steinberg representation sees the correct fixed-space dimension but cannot see any of the finite indices above. The Smith factors record the integral refinement: they measure how far averaging and transfer are from being unimodular on the chosen homology lattice.

7.2 The norm map

For a finite group H acting on a lattice A, define the norm by

$$N_H : A_H \longrightarrow A^H, \qquad N_H([a]) = \sum_{h \in H} ha.$$

Here $[a]$ is the coinvariant class of a. The sum is invariant and vanishes on every relation $ha - a$, so the map is well-defined. Its kernel and cokernel are, by definition, the Tate cohomology groups $\widehat{H}^{-1}(H, A)$ and $\widehat{H}^0(H, A)$, respectively [4, Section 6.3].

Corollary 7.2. *For the Singer–Frobenius action,*

$q \bmod 3$	0	1	2
$\widehat{H}^{-1}(\Gamma_q, M_q)$	0	0	$\mathbb{Z}/3$
$\widehat{H}^0(\Gamma_q, M_q)$	0	$\mathbb{Z}/9$	0

Proof. Write $n = |\Gamma_q| = 3m$. Since the invariant lattice is torsion-free, the norm kills the torsion of the coinvariants and induces $\bar{N} : Q_q \to M_q^{\Gamma_q}$. On invariants the norm is multiplication by n, and in coinvariants every translate has the same class. Consequently

$$\bar{N} j_q = n\,\mathrm{id}_{M_q^{\Gamma_q}}, \qquad j_q \bar{N} = n\,\mathrm{id}_{Q_q}.$$

In particular $\bar{N}$ is injective, so the kernel of the original norm is exactly the torsion computed in Theorem 1.1. Choose bases in which j_q has Smith diagonal $d_1, \dots, d_s$, where s is the common rank. The first identity forces the norm matrix, in the corresponding bases, to be $\mathrm{diag}(n/d_1, \dots, n/d_s)$. Theorem 7.1 gives only entries n when $q \equiv 0, 2 \pmod 3$, and the entries $m/3, n, \dots, n$ when $q \equiv 1 \pmod 3$. The norm diagonal is therefore the identity in the first two cases and $9, 1, \dots, 1$ in the last case. This proves the asserted cokernels, including rank zero when $q = 2$. □

For $q = 4$, for instance, $m = 21$ and the comparison diagonal is $7, 63$, while the corresponding norm diagonal is $9, 1$. Thus the index-three projection defect in Section 6 and the order-nine norm cokernel describe different maps. The corollary is a consequence of the comparison theorem.

Remark 7.3. The geometry and the transfer for $X_q \to X_q/K$ are classical. The calculation concerns the specified integral homology lattice restricted to Γ_q.

8 A composite-rank obstruction

For higher rank, let $r \geq 2$, put $V = \mathbb{F}_{q^r}$ as an r-dimensional vector space over $\mathbb{F}_q$, and let $X_{r,q}$ be the order complex of the nonzero proper subspaces of V. Write

$$M_{r,q} = \widetilde{H}_{r-2}(X_{r,q}; \mathbb{Z}), \qquad K_{r,q} = \mathbb{F}_{q^r}^\times / \mathbb{F}_q^\times, qquad \Gamma_{r,q} = K_{r,q} \rtimes \langle \sigma \rangle,$$

where $\sigma(x) = x^q$. The case studied in this note is $(r, q) = (3, q)$.

First consider the case in which r is prime. The following elementary fact about Tate cohomology will be used below.

Lemma 8.1 (Prime-to-ℓ reduction)**.** *Let $1 \to N \to G \to Q \to 1$ be an extension of finite groups, let A be a G-lattice, and let $\ell \nmid |Q|$. Then, for every integer j,*

$$\widehat{H}^j(G, A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)} \cong \left(\widehat{H}^j(N, A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)}\right)^Q.$$

The action of Q on the right is the one induced by conjugation on N and by the given action on A.

Proof. Let res be restriction from G to N, and let cor be the reverse transfer. Restriction lands in the Q-invariants. The two maps satisfy

$$\operatorname{cor}\operatorname{res} = |Q|\,\mathrm{id}, \qquad \operatorname{res}\operatorname{cor} = \sum_{\bar{g} \in Q} \bar{g}.$$

After localization at ℓ, $|Q|$ is invertible. The first identity therefore makes restriction injective. If x is Q-invariant, then $|Q|^{-1}\operatorname{cor}(x)$ restricts to x by the second identity. Thus restriction is an isomorphism onto the invariant part. The two identities are the usual restriction–corestriction formulas for Tate cohomology; see [4, Sections 5.2 and 6.3]. □

Theorem 8.2 (Prime-rank vanishing)**.** *If r is prime, then for every prime $\ell \nmid r$,*

$$\widehat{H}^{-1}(\Gamma_{r,q}, M_{r,q}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)} = 0, \qquad \widehat{H}^0(\Gamma_{r,q}, M_{r,q}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)} = 0.$$

Proof. Let $m = (q^r - 1)/(q - 1)$. If $1 \neq a \in K_{r,q}$ fixed a proper nonzero $\mathbb{F}_q$-subspace W projectively, then for some $c \in \mathbb{F}_q^\times$ the element $c^{-1}a$ preserves W. When r is prime, $a \notin \mathbb{F}_q$ implies $\mathbb{F}_q(a) = \mathbb{F}_{q^r}$, so W would be an $\mathbb{F}_{q^r}$-subspace. This is impossible. Thus $K_{r,q}$ acts freely on every simplex of $X_{r,q}$.

The augmented building chain complex has the form

$$0 \longrightarrow M_{r,q} \longrightarrow C_{r-2} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

with each C_i a free $\mathbb{Z}[K_{r,q}]$-module. Dimension shifting gives

$$\widehat{H}^j(K_{r,q}, M_{r,q}) \cong \widehat{H}^{j-r+1}(K_{r,q}, \mathbb{Z}).$$

Indeed, the Tate cohomology of each free C_i is zero, so each short exact sequence in the displayed chain complex shifts the degree by one. For the cyclic group $K_{r,q} \cong C_m$, the right side is $\mathbb{Z}/m$ in even degrees and zero in odd degrees. If $r = 2$, this gives

$$\widehat{H}^{-1}(K_{2,q}, M_{2,q}) \cong \mathbb{Z}/(q + 1), \qquad \widehat{H}^0(K_{2,q}, M_{2,q}) = 0.$$

Frobenius acts on the first group by multiplication by q^{-1}, which is -1 modulo $q + 1$; its invariant part is therefore 2-primary.

Now let $r = 2s + 1$ be odd. Then

$$\widehat{H}^{-1}(K_{r,q}, M_{r,q}) = 0, \qquad \widehat{H}^0(K_{r,q}, M_{r,q}) \cong \mathbb{Z}/m,$$

and the periodic cyclic resolution identifies the Frobenius action on the last group with multiplication by q^{-s}. If a prime ℓ divides both m and $q^s - 1$, then either $\ell \mid q - 1$, in which case $m \equiv r \pmod{\ell}$, or the order of q modulo ℓ divides both r and s. The latter is impossible, and the former implies $\ell \mid r$. Hence the C_r-invariants vanish after localizing at any $\ell \nmid r$. Finally, apply Lemma 8.1 to $1 \to K_{r,q} \to \Gamma_{r,q} \to C_r \to 1$. Since $\ell \nmid r$, the required C_r-invariant parts vanish after localization, proving both assertions. □

For composite rank, a Sylow subgroup can fix a genuine subbuilding. The next lemma turns that fixed subbuilding into a Tate-cohomology calculation.

Lemma 8.3 (Tate shift across a fixed subbuilding)**.** *Let a cyclic ℓ-group P act simplicially on a finite complex X, with the vertex ordering preserved by the action. Assume that every simplex is either fixed pointwise by P or has trivial stabilizer, and write $Y = X^P$. Suppose that the reduced homology of X is concentrated in degree n, with lattice M, and that the reduced homology of Y is concentrated in degree b, with lattice M'. Then*

$$\widehat{H}^j(P, M) \cong \widehat{H}^{j-n+b}(P, M')$$

for every j. The isomorphism is equivariant for every group of simplicial automorphisms which normalizes P and preserves Y.

Proof. Let $\widetilde{C}_*(X)$ and $\widetilde{C}_*(Y)$ be the augmented reduced chain complexes. Their inclusion gives

$$0 \longrightarrow \widetilde{C}_*(Y) \longrightarrow \widetilde{C}_*(X) \longrightarrow Q_* \longrightarrow 0.$$

Every simplex represented in Q_* has a free P-orbit, so each Q_i is a free $\mathbb{Z}[P]$-module. Free modules have zero Tate cohomology, and hence the bounded complex Q_* contributes nothing to the long exact Tate sequence. The remaining complex $\widetilde{C}_*(X)$ has only one nonzero homology group, namely M in degree n; replacing it by that homology group shifts the Tate degree by n. The same argument for Y shifts it by b. Comparing the two complexes gives $\widehat{H}^j(P, M) \cong \widehat{H}^{j-n+b}(P, M')$. All maps are induced by chain maps, so automorphisms normalizing P act equivariantly on the isomorphism. □

Apply the lemma in rank six.

Theorem 8.4 (A prime-to-r counterexample)**.** *For $(r, q) = (6, 4)$ and $\Gamma = \Gamma_{6,4}$,*

$$\widehat{H}^{-1}(\Gamma, M_{6,4}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(5)} \neq 0.$$

In particular, the prime-to-r norm-vanishing statement is false in composite rank: $5 \nmid 6$, but the norm has nontrivial 5-primary kernel.

Proof. The group $K_{6,4}$ has order $(4^6 - 1)/(4 - 1) = 1365$. Let $P \leq K_{6,4}$ have order five. It is the image of $\mathbb{F}_{16}^\times/\mathbb{F}_4^\times$. A nonidentity element of P generates $\mathbb{F}_{16}$ over $\mathbb{F}_4$. If such an element fixes an $\mathbb{F}_4$-subspace projectively, then that subspace is stable under $\mathbb{F}_{16}$ and is therefore an $\mathbb{F}_{16}$-subspace. Thus

$$X_{6,4}^P = X_{3,16}.$$

The action preserves dimensions, so a stabilized simplex has every vertex fixed. Every simplex outside $X_{3,16}$ therefore has trivial stabilizer. The fixed-subbuilding lemma, with $(n, b) = (4, 1)$, gives

$$\widehat{H}^{-1}(P, M_{6,4}) \cong \widehat{H}^{-4}(P, M_{3,16}) \cong M_{3,16}/5M_{3,16}.$$

Here P fixes the subbuilding pointwise, and the two-periodic resolution of C_5 gives the last identification. Conjugation by σ sends a generator of P to its fourth power. In degree -4 this contributes the scalar 4^{-2}, which is 1 modulo five. Hence the induced action on $M_{3,16}/5M_{3,16}$ is the ordinary Frobenius action.

Set $S = K_{6,4}/P \cong K_{3,16}$. Its order is 273, so averaging over S is valid modulo five. The Singer action of S on $X_{3,16}$ is free, and transfer identifies

$$A := M_{3,16}^S \cong H_1(X_{3,16}/S; \mathbb{Z}) \cong \ker\big(\text{aug} : \mathbb{Z}^{17} \longrightarrow \mathbb{Z}\big).$$

The quotient graph has two vertices and seventeen edges. Let $\tau = \sigma^2$, the 16-Frobenius. Its action on these edges has two fixed edges and five three-cycles, so for $E = \mathbb{Z}^{17}$,

$$E^\tau \cong \mathbb{Z}^7, \qquad A^\tau = \ker(\text{aug} : E^\tau \to \mathbb{Z}).$$

The seven orbit-sums form a basis of E^τ. Since $\sigma^2 = \tau$, σ induces an involution on this basis and hence has at least four orbits. The sum of the basis vectors in any such orbit has augmentation 1, 2, 3, or 6, all nonzero modulo five. The augmentation map on the σ-fixed subspace is therefore onto $\mathbb{F}_5$, and

$$\dim_{\mathbb{F}_5}(A/5A)^{\langle\sigma\rangle} \geq 4 - 1 = 3.$$

The prime-to-5 reduction lemma applied to $P \triangleleft \Gamma$ now identifies the 5-local Tate group with the (Γ/P)-invariants of $M_{3,16}/5M_{3,16}$. Averaging over S replaces these by $A/5A$, and the remaining generator is σ. Hence

$$\widehat{H}^{-1}(\Gamma, M_{6,4}) \otimes \mathbb{Z}_{(5)} \cong (A/5A)^{\langle\sigma\rangle} \neq 0.$$

□

Corollary 8.5 (A rank-six family of obstructions)**.** *Let ℓ be an odd prime with $\ell \neq 3$ and $\ell \mid (q+1)$. Then*

$$\widehat{H}^{-1}(\Gamma_{6,q}, M_{6,q}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\ell)} \neq 0.$$

More precisely, put

$$b_q = \begin{cases} (q^2+3)/3, & 3 \mid q, \\ (q^2+5)/3, & 3 \nmid q. \end{cases}$$

Then

$$\dim_{\mathbb{F}_\ell}\left(\widehat{H}^{-1}(\Gamma_{6,q}, M_{6,q}) \otimes_{\mathbb{Z}} \mathbb{F}_\ell\right) \geq \lceil b_q/2 \rceil - 1.$$

The preceding theorem is the case $(q, \ell) = (4, 5)$.

Proof. Because ℓ is odd and $\ell \mid q+1$, the order of q modulo ℓ is two. Let P be the Sylow ℓ-subgroup of $K_{6,q}$, and put

$$R = \mathbb{F}_{q^2}^\times / \mathbb{F}_q^\times \leq K_{6,q}.$$

Then $P \leq R$, although R/P may be nontrivial. A nonidentity element of P generates $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. The argument used above therefore gives

$$X_{6,q}^P = X_{3,q^2}, \qquad \widehat{H}^{-1}(P, M_{6,q}) \cong \widehat{H}^{-4}(P, M_{3,q^2}).$$

The action of σ on the last Tate group has the scalar q^{-2} in addition to its action on M_{3,q^2}. Since $q^2 \equiv 1 \pmod{\ell}$, this scalar disappears after reduction modulo ℓ. The quotient $K_{6,q}/P$ acts on the fixed subbuilding with kernel R/P and image

$$K_{6,q}/R \cong K_{3,q^2},$$

the Singer group of the fixed subbuilding. Since P acts trivially there, the two-periodic resolution gives $\widehat{H}^{-4}(P, M_{3,q^2}) = M_{3,q^2}/|P|M_{3,q^2}$. Its reduction modulo ℓ is $M_{3,q^2}/\ell M_{3,q^2}$. Averaging over $K_{6,q}/P$ therefore reduces the invariant calculation to

$$A := M_{3,q^2}^{K_{3,q^2}} \cong \ker(\text{aug} : \mathbb{Z}^{q^2+1} \to \mathbb{Z}).$$

The square $\tau = \sigma^2$ is the q^2-Frobenius. On the q^2+1 edges of the quotient graph it has one fixed edge and $q^2/3$ three-cycles when $3 \mid q$, and two fixed edges and $(q^2-1)/3$ three-cycles otherwise. Thus the τ-orbit sums form a lattice of rank b_q, and A^τ is their augmentation kernel. The relation $\sigma^2 = \tau$ makes σ an involution on the b_q orbit sums, so it has at least $\lceil b_q/2 \rceil$ orbits. The invariant sum of each

orbit has augmentation 1, 2, 3, or 6. Since $\ell \neq 2, 3$, these are nonzero modulo ℓ. The augmentation map on the invariant subspace is therefore surjective, and

$$\dim_{\mathbb{F}_\ell}(A/\ell A)^{\langle\sigma\rangle} \geq \lceil b_q/2 \rceil - 1.$$

Since $|\Gamma_{6,q}/P|$ is prime to ℓ, averaging commutes with reduction modulo ℓ. The prime-to-ℓ reduction lemma therefore identifies this invariant space with the reduction of the ℓ-primary part of $\widehat{H}^{-1}(\Gamma_{6,q}, M_{6,q})$. This proves the bound and the nonvanishing statement. □

The obstruction comes from the lower-rank lattice on the fixed subbuilding. Any composite-rank vanishing statement must either control the prime-to-ℓ Singer quotient or retain these fixed-subbuilding terms.

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